

New Coherent String States and Minimal Uncertainty in WZWN Models

A.L. Larsen¹ and N. Sánchez²

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¹Department of Physics, University of Odense, Campusvej 55, 5230 Odense M, Denmark.

²Observatoire de Paris, DEMIRM. Laboratoire Associé au CNRS, Observatoire de Paris et École Normale Supérieure. 61, Avenue de l'Observatoire, 75014 Paris, France.

Abstract

We study the properties of **exact** (all level k) quantum coherent states in the context of string theory on a group manifold (WZWN models). Coherent states of WZWN models may help to solve the unitarity problem: Having positive norm, they consistently describe the very massive string states (otherwise excluded by the spin-level condition). These states can be constructed by (at least) two alternative procedures: (i) as the exponential of the creation operator on the ground state, and (ii) as eigenstates of the annihilation operator. In the $k \rightarrow \infty$ limit, all the known properties of ordinary coherent states are recovered. States (i) and (ii) (which are equivalent in the context of ordinary quantum mechanics and string theory in flat spacetime) are not equivalent in the context of WZWN models. The set (i) was constructed by these authors in a previous article. In this paper we provide the construction of states (ii), we compare the two sets and discuss their properties. We analyze the uncertainty relation, and show that states (ii) satisfy automatically the *minimal uncertainty* condition for any k ; they are thus *quasiclassical*, in some sense more classical than states (i) which only satisfy it in the $k \rightarrow \infty$ limit. Modification to the Heisenberg relation is given by $2\mathcal{H}/k$, where \mathcal{H} is connected to the string energy.

1 Introduction

Coherent states play an important role in quantum mechanics, where they represent the "quasi-classical" states of minimal uncertainty (see for instance [1]). Coherent states have also been widely used in string theory for the computation of scattering amplitudes (see for instance [2]). Both in ordinary quantum mechanics and in the standard formulation of string theory, coherent states can be defined in (at least) two alternative but *equivalent* ways: Either as eigenstates of the annihilation operator, or as the exponential of the creation operator acting on the ground state. However, when considering string theory on a group manifold using a WZWN or gauged WZWN construction, the situation is quite different. In that case, the fundamental Abelian harmonic oscillator commutator $[a, a^\dagger] = 1$ is substituted by a non-Abelian Kac-Moody current algebra. As a consequence, a state defined as the exponential of a creation operator acting on the ground state, will no longer in general be an eigenstate of the annihilation operator. For the same reason, it will generally not be a state of minimal uncertainty.

The purpose of the present paper is to discuss the alternative, but *inequivalent*, definitions of coherent states in WZWN models. More precisely, we shall consider the $SL(2, R)$ WZWN model corresponding to bosonic string theory in 3-dimensional Anti de Sitter space, $AdS_3 \cong SL(2, R) \cong SU(1, 1)$. It represents the simplest example of string theory on a manifold with curved space *and* curved time [3-16], and it has attracted renewed interest recently in the context of the conjecture [17] connecting supergravity and superstring theory in AdS space with a conformal field theory on the boundary. For a recent discussion about coherent states on group manifolds, see also [18].

In a previous publication [19], we considered coherent states in AdS_3 using the definition corresponding to the exponential of a creation operator. Such states were shown to describe, among other things, the very massive string states in AdS_3 . In particular, it was shown that there is a discrete spectrum of very massive string states, with asymptotic behaviour $m^2 \alpha' \propto N^2$ (N positive integer). This was in precise agreement with the previous results obtained using semi-classical quantization [20, 21, 22], and the same asymptotic behaviour was also obtained in ref.[23], although the construction there was completely different from ours.

The coherent states, defined in terms of the exponential of a creation operator, however, are not eigenstates of the annihilation operator, and they

are not minimal uncertainty states. Moreover, they are somewhat complicated to work with. In this paper we consider the alternative definition of coherent states in AdS_3 , taking the property of being an eigenstate of the annihilation operator as the fundamental one. Such states will also be automatically states of minimal uncertainty, and they are generally much easier to work with.

There is an extensive literature on coherent states of various kinds; canonical coherent states, spin coherent states, group-related coherent states etc. (for a review see [24]). The coherent states constructed in this paper generalize the $SU(1, 1)$ group-related coherent states originally constructed in [25]. The coherent states in [25] were constructed using the ladder-operators. The ladder-operators correspond to the zero-modes of the Kac-Moody algebra: $L^\pm = \frac{1}{\sqrt{2}}J_0^\pm$, $L_{12} = J_0^3$. In string theory, the zero-modes are not really creation and annihilation operators. Therefore, we use the $n = 1$ modes instead. So our coherent states are completely different from theirs. However, it is interesting that our results reduce to theirs if we take $k = 0$, where k is the level of the WZW model. The reason is that the $n = 1$ algebra, for $k = 0$, is formally identical to the zero-mode algebra. Thus, all our results for $k = 0$ reduce to theirs.

This paper is organized as follows. In Section 2, we first review the standard formulation of string theory on a group manifold [26, 27]. We then derive the explicit expression for the coherent states defined as eigenstates of the annihilation operator. Normalization, Virasoro and mass-shell conditions are also considered. In Section 3, we show that the coherent states constructed in Section 2 are states of minimal uncertainty, in the usual sense of ordinary quantum mechanics. In Section 4, we consider the relation between these "new" coherent states and the "old" coherent states discussed in [19]. Finally in Section 5, we have some concluding remarks.

2 Coherent States

The $SL(2, R)$ Kac-Moody algebra for (say) the left-moving currents is given by

$$[J_m^a, J_n^b] = i\epsilon^{ab}{}_c J_{m+n}^c + \frac{k}{2} m\eta^{ab} \delta_{n+m} \quad (2.1)$$

where

$$k = \frac{1}{H^2 \alpha'} ,$$

is the level of the $SL(2, R)$ WZWN model, and H^{-1} stands for the length scale. (Our conventions are: $\eta^{ab} = \text{diag}(1, 1, -1)$ and $\epsilon^{123} = +1$).

In terms of the currents, $J^\pm = J^1 \pm iJ^2$, the algebra becomes

$$\begin{aligned} [J_m^+, J_n^-] &= -2J_{m+n}^3 + km\delta_{m+n} \\ [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm \\ [J_m^3, J_n^3] &= -\frac{k}{2}m\delta_{m+n} \end{aligned} \quad (2.2)$$

The world-sheet energy-momentum tensor takes the Sugawara form

$$T = \frac{1}{k-2} \eta_{ab} : J^a J^b : = \frac{1}{k-2} : (J^+ J^- - J^3 J^3) : \quad (2.3)$$

Its Fourier modes

$$T = \sum_{n=-\infty}^{\infty} L_n e^{-in\sigma} \quad (2.4)$$

are given by

$$L_n = \frac{1}{k-2} \sum_{l=-\infty}^{\infty} : \left(\frac{1}{2} (J_{n-l}^+ J_l^- + J_{n-l}^- J_l^+) - J_{n-l}^3 J_l^3 \right) : \quad (2.5)$$

They fulfill the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \quad (2.6)$$

where the central charge is given by

$$c = \frac{3k}{k-2} \quad (2.7)$$

Demanding $c = 26$, corresponding to conformal invariance, gives $k = 52/23$. Notice also the commutators

$$[L_n, J_m^\pm] = -mJ_{n+m}^\pm, \quad [L_n, J_m^3] = -mJ_{n+m}^3 \quad (2.8)$$

which will be useful in the following.

The Kac-Moody algebra contains the subalgebra of zero modes J_0^a , for which the quadratic Casimir is

$$Q = \eta_{ab} J_0^a J_0^b = \frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3 \quad (2.9)$$

The primary states, which are quantum states $|jm\rangle$ at grade zero ("base-states" or "ground-states"), are characterised by

$$Q|jm\rangle = -j(j+1)|jm\rangle, \quad J_0^3|jm\rangle = m|jm\rangle \quad (2.10)$$

Moreover, they fulfill

$$J_0^\pm |jm\rangle = \sqrt{m(m \pm 1) - j(j+1)} |jm \pm 1\rangle \quad (2.11)$$

as well as

$$J_l^a |jm\rangle = 0; \quad l > 0 \quad (2.12)$$

The primary states must belong to one of the unitary representations of $SL(2, R)$ (or its covering group) [4, 28].

For simplicity and clarity of the construction, we concentrate in the following on the subalgebra generated by $(J_{+1}^-, J_{-1}^+, J_0^3)$

$$\begin{aligned} [J_{+1}^-, J_{-1}^+] &= 2J_0^3 + k \\ [J_0^3, J_{-1}^+] &= J_{-1}^+ \\ [J_0^3, J_{+1}^-] &= -J_{+1}^- \end{aligned} \quad (2.13)$$

An example of string configurations described by this subalgebra is provided by circular strings (which contain only modes corresponding to $n = 0$ and $n = \pm 1$); it should be stressed that we consider this subalgebra only for simplicity and clarity and that our construction can be easily used for other string configurations as well.

Moreover, our coherent states will be constructed using the base-state $|jj\rangle$, which belongs to the highest weight discrete series D_j^- [4, 28], with states $|jm\rangle$

$$j \leq -1/2, \quad m = j, j-1, \dots \quad (2.14)$$

Since we shall consider the covering group of $SL(2, R)$, there are no further restrictions on j , i.e., it does not need to be an integer or half-integer [4, 28]. In particular, from eq.(2.11) it follows that

$$J_0^+ |jj\rangle = 0, \quad J_0^- |jj\rangle = \sqrt{-2j} |jj-1\rangle \quad (2.15)$$

The idea is now to construct coherent states as eigenstates of the annihilation operator J_{+1}^-

$$J_{+1}^- |\mu\rangle = \mu |\mu\rangle \quad (2.16)$$

where μ is a complex number. For the state $|\mu\rangle$ we use the ansatz

$$|\mu\rangle = \mathcal{N} \sum_{n=0}^{\infty} C_n (J_{-1}^+)^n |jj\rangle \quad (2.17)$$

where \mathcal{N} is a normalization constant, and the coefficients C_n are to be determined. Using the commutator

$$[J_{+1}^-, (J_{-1}^+)^n] = n (J_{-1}^+)^{n-1} (n-1+k+2J_0^3) \quad (2.18)$$

eq.(2.16) immediately leads to the recursion relation

$$(n+1)(n+2j+k)C_{n+1} = \mu C_n \quad (2.19)$$

which is solved by

$$C_n = \mu^n \frac{\Gamma(2j+k)}{\Gamma(n+1)\Gamma(2j+k+n)} \quad (2.20)$$

with the normalization $C_0 = 1$.

Using the identity

$$\langle jj | (J_{+1}^-)^n (J_{-1}^+)^m | jj \rangle = \delta_{nm} \frac{\Gamma(n+1)\Gamma(2j+k+n)}{\Gamma(2j+k)} \quad (2.21)$$

the normalization condition for the state $|\mu\rangle$ leads to

$$1 = \langle \mu | \mu \rangle = |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{(\mu^* \mu)^n}{n!} \frac{\Gamma(2j+k)}{\Gamma(2j+k+n)} \quad (2.22)$$

That is to say

$$|\mathcal{N}|^{-2} = \Gamma(2j+k) |\mu|^{-(2j+k-1)} I_{2j+k-1}(2|\mu|) \quad (2.23)$$

where $I_\nu(x)$ is the modified Bessel function [29] and $|\mu|^2 = \mu^* \mu$. To ensure that the right hand side of eq.(2.23) is positive for arbitrary complex μ , we take $2j+k$ positive. Thus we get the spin-level restriction [4-6,8-14]

$$j > -\frac{k}{2} \quad (2.24)$$

Therefore, altogether, for $2j + k$ positive, we have

$$\begin{aligned} |\mu\rangle &= \mathcal{N} \sum_{n=0}^{\infty} \mu^n \frac{\Gamma(2j+k)}{\Gamma(n+1)\Gamma(2j+k+n)} (J_{-1}^+)^n |jj\rangle \\ &= \mathcal{N} \Gamma(2j+k) (\mu J_{-1}^+)^{-(2j+k-1)/2} I_{2j+k-1} \left(2\sqrt{\mu J_{-1}^+} \right) |jj\rangle \end{aligned} \quad (2.25)$$

where \mathcal{N} is given by eq.(2.23). As in quantum mechanics, the coherent states (2.25) do not form an orthogonal set. The scalar product of two coherent states is given by

$$|\langle \nu | \mu \rangle|^2 = \frac{I_{2j+k-1}(2\sqrt{\nu^* \mu}) I_{2j+k-1}(2\sqrt{\mu^* \nu})}{I_{2j+k-1}(2\sqrt{\nu^* \nu}) I_{2j+k-1}(2\sqrt{\mu^* \mu})} \quad (2.26)$$

In string theory, a physical state must fulfill the mass-shell condition and the Virasoro primary conditions

$$(L_0 - 1)|\psi\rangle = 0, \quad L_l|\psi\rangle = 0; \quad l > 0 \quad (2.27)$$

For the coherent states (2.25) it is easy to see that the Virasoro primary conditions are fulfilled. However, being coherent states, they obviously are not eigenstates of neither the number operator nor of the L_0 operator. We shall therefore impose a "weak" mass-shell condition

$$\langle \mu | (L_0 - 1) | \mu \rangle = 0 \quad (2.28)$$

Using the identity

$$L_0 (J_{-1}^+)^n |jj\rangle = \left(n - \frac{j(j+1)}{k-2} \right) (J_{-1}^+)^n |jj\rangle \quad (2.29)$$

the condition (2.28) leads to

$$\frac{I_{2j+k}(2|\mu|)}{I_{2j+k-1}(2|\mu|)} = |\mu|^{-1} \left(1 + \frac{j(j+1)}{k-2} \right) \quad (2.30)$$

which is to be solved for (say) j as a function of μ .

We close this section with some comments on the case where $2j + k = \{0, -1, -2, \dots\} = -N$, where N is a non-negative integer. In that case the

solution (2.20) for C_n is actually not well-defined. Instead the recursion relation (2.19) is solved by

$$\begin{cases} C_n = 0; & n = 0, 1, \dots, N \\ C_{N+1+l} = \frac{\mu^l}{l!} \frac{\Gamma(N+2)}{\Gamma(N+2+l)}; & l \geq 0 \end{cases} \quad (2.31)$$

and the coherent state is given by

$$\begin{aligned} |\mu\rangle &= \mathcal{N} \sum_{n=0}^{\infty} \mu^n \frac{\Gamma(N+2)}{\Gamma(n+1)\Gamma(N+2+n)} (J_{-1}^+)^{N+1+n} |-(N+k)/2, -(N+k)/2\rangle \\ &= \mathcal{N} \frac{\Gamma(N+2)}{\mu^{N+1}} (\mu J_{-1}^+)^{(N+1)/2} I_{N+1} \left(2\sqrt{\mu J_{-1}^+} \right) |-(N+k)/2, -(N+k)/2\rangle \end{aligned} \quad (2.32)$$

where \mathcal{N} is a normalization constant. However, using the identity

$$\begin{aligned} \langle -(N+k)/2, -(N+k)/2 | (J_{-1}^-)^n (J_{-1}^+)^m |-(N+k)/2, -(N+k)/2\rangle \\ = (-1)^n \delta_{nm} n! \begin{cases} \frac{N!}{(N-n)!}; & n \leq N \\ 0; & n > N \end{cases} \end{aligned} \quad (2.33)$$

one finds that $\langle \mu | \mu \rangle = 0$. That is to say, the coherent states for $2j+k = -N$ are zero norm states, so we shall not consider them further.

We are thus left with a continuous spectrum of the positive norm coherent states (2.25). The discrete spectrum of states (2.32) are all zero norm states, and are expected to decouple in scattering amplitudes.

3 Minimal Uncertainty

One of the most important properties of coherent states in quantum mechanics is the one of minimal uncertainty [1]

$$\Delta X \cdot \Delta P = \frac{1}{2} \quad (3.1)$$

We shall now show that the states (2.25) lead to the same property in the case of a Kac-Moody algebra. First we define Hermitean operators (X, P, \mathcal{H})

$$X \equiv \frac{1}{\sqrt{2k}} (J_{+1}^- + J_{-1}^+)$$

$$\begin{aligned}
P &\equiv \frac{-i}{\sqrt{2k}}(J_{+1}^- - J_{-1}^+) \\
\mathcal{H} &\equiv J_0^3
\end{aligned} \tag{3.2}$$

Then, the algebra (2.13) becomes

$$\begin{aligned}
[X, P] &= i(1 + \frac{2}{k}\mathcal{H}) \\
[X, \mathcal{H}] &= iP \\
[\mathcal{H}, P] &= iX
\end{aligned} \tag{3.3}$$

The algebra (3.3) can be interpreted as a modified Harmonic oscillator algebra; the modification being represented by the second term $\frac{2}{k}\mathcal{H}$ in the X, P commutator. That is, in the semi-classical limit ($k \rightarrow \infty$), we get the standard Harmonic oscillator algebra [1]. It is quite natural to interpret X and P as coordinate and momentum, respectively, since, in the context of Kac-Moody algebras, the roles of coordinates and momenta are played by the currents J . However, the interpretation of \mathcal{H} as some kind of Hamiltonian needs a few comments: First, notice that, contrary to the case of the standard harmonic oscillator, \mathcal{H} is here an *independent* operator; in particular, $\mathcal{H} \neq \frac{1}{2}(P^2 + X^2)$. On the other hand, there is a simple relation between J_0^3 and the energy E and angular momentum l of a string in AdS_3 [19, 20, 23]

$$\begin{aligned}
J_0^3 &= \frac{1}{2\pi} \int_0^{2\pi} J^3 d\sigma = \frac{1}{2}(E + l) \\
\bar{J}_0^3 &= \frac{1}{2\pi} \int_0^{2\pi} \bar{J}^3 d\sigma = \frac{1}{2}(E - l)
\end{aligned} \tag{3.4}$$

where a bar denotes right-movers. Thus, the total energy is $E = J_0^3 + \bar{J}_0^3$ and it is natural to identify $\mathcal{H} \sim J_0^3$.

From eqs.(3.3), the uncertainty relation here is given by

$$\Delta X \cdot \Delta P \geq \frac{1}{2} | \langle (1 + \frac{2}{k}\mathcal{H}) \rangle | \tag{3.5}$$

with minimal uncertainty in the case of equality sign. For $k \rightarrow \infty$, it is the usual Heisenberg relation.

Now, consider the coherent states (2.25). It is straightforward to compute

$$(\Delta X)^2 = (\Delta P)^2 = \frac{1}{2k} \left(k + 2(j+1)(1 + \frac{j}{k-2}) \right) \tag{3.6}$$

as well as

$$\langle \mathcal{H} \rangle = (j+1)\left(1 + \frac{j}{k-2}\right) \quad (3.7)$$

That is to say

$$\Delta X \cdot \Delta P = \frac{1}{2} \left| \langle (1 + \frac{2}{k} \mathcal{H}) \rangle \right| \quad (3.8)$$

i.e., minimal uncertainty. That is to say, states (2.25) are quasiclassical states.

4 "Exponential" Coherent States

In a previous paper [19], we considered a different type of coherent states defined in terms of the exponential of the creation operator

$$e^{\tilde{\mu} J_{-1}^+} |jj\rangle \quad (4.1)$$

where $\tilde{\mu}$ is an arbitrary complex number. Such coherent states (4.1) are however not eigenstates of the annihilation operator J_{+1}^-

$$J_{+1}^- e^{\mu J_{-1}^+} |jj\rangle = \mu (2j + k + \mu J_{-1}^+) e^{\mu J_{-1}^+} |jj\rangle \quad (4.2)$$

As for the normalization of states (4.1), we use the identity

$$\langle jj | e^{\tilde{\mu}^* J_{+1}^-} e^{\tilde{\mu} J_{-1}^+} |jj\rangle = 1 + \sum_{n=1}^{\infty} \frac{(\tilde{\mu}^* \tilde{\mu})^n}{n!} \prod_{l=1}^n (2j + k - 1 + l) \quad (4.3)$$

The product on the right hand side goes as $n!$. Thus the infinite sum is convergent only if $\tilde{\mu}^* \tilde{\mu} < 1$, or if the infinite sum terminates after a finite number of terms (this happens if $2j + k - 1 + l = 0$, for some l). More precisely, the right hand side of eq.(4.3) is a finite positive number in the following two cases

(I): $\tilde{\mu}^* \tilde{\mu} < 1$ and j arbitrary ($j \leq -1/2$).

In this case the normalized state is

$$|\tilde{\mu}_I\rangle = (1 - \tilde{\mu}^* \tilde{\mu})^{j+k/2} e^{\tilde{\mu} J_{-1}^+} |jj\rangle \quad (4.4)$$

(II): $\tilde{\mu}^* \tilde{\mu} > 1$ and $2j + k = -N$ ($N = 0, 1, 2, \dots$).

In this case the normalized state is

$$|\tilde{\mu}_{II}\rangle = (\tilde{\mu}^* \tilde{\mu} - 1)^{-N} e^{\tilde{\mu} J_{-1}^+} | -N - k/2, -N - k/2 \rangle \quad (4.5)$$

The Virasoro primary conditions are obviously fulfilled for the states (4.4)-(4.5), while the mass-shell condition in the form of eq.(2.28) gives rise to some additional constraints on $\tilde{\mu}$ and j . In the two cases one finds, respectively

(I):

$$\tilde{\mu}^* \tilde{\mu} = \frac{1 + \frac{j(j+1)}{k-2}}{2j + k + 1 + \frac{j(j+1)}{k-2}} < 1 ; \quad -\frac{k}{2} < j \leq -\frac{1}{2} \quad (4.6)$$

(II):

$$\tilde{\mu}^* \tilde{\mu} = \frac{1 + \frac{j(j+1)}{k-2}}{2j + k + 1 + \frac{j(j+1)}{k-2}} > 1 ; \quad j = -N - \frac{k}{2} \quad (N = 1, 2, \dots) \quad (4.7)$$

It follows that the spectrum consists of two parts [19]: **(I)** A continuous spectrum where j fulfills the standard spin-level condition [4-6,8-14] $-k/2 < j \leq -1/2$, and **(II)** a discrete spectrum where j fulfills $j = -N - k/2$ (N positive integer). The discrete spectrum describes very massive string states, with asymptotic behaviour $m^2 \alpha' \propto N^2$ (N positive integer) [19]. This is in precise agreement with previous results obtained using semi-classical quantization [20, 21, 22], and the same asymptotic behaviour was also obtained in the recent paper [23].

Unfortunately, the states (4.4)-(4.5) are somewhat complicated to work with since they are not eigenstates of the annihilation operator. Therefore it would be useful to express them in terms of the states (2.25). More generally, let us consider the off-shell relationship between states (2.25) and states (4.4). Clearly, the two types of coherent states are not orthogonal

$$\langle \mu | \tilde{\mu}_I \rangle = \frac{(1 - \tilde{\mu}^* \tilde{\mu})^{j+k/2} |\mu|^{(2j+k-1)/2} e^{\mu^* \tilde{\mu}}}{\sqrt{\Gamma(2j+k) I_{2j+k-1}(2|\mu|)}} \quad (4.8)$$

but it is possible to express (say) the states $|\tilde{\mu}_I \rangle$ in terms of the states $|\mu \rangle$. However, since the states $|\mu \rangle$ form an over-complete set, the expression will of course not be unique. So, we just give an example:

For fixed $2j + k$, we first introduce normalized basis states $|n\rangle$

$$|n\rangle = \sqrt{\frac{\Gamma(2j+k)}{\Gamma(n+1)\Gamma(2j+k+n)}} \left(J_{-1}^+\right)^n |jj\rangle; \quad \langle n|m\rangle = \delta_{nm} \quad (4.9)$$

It follows that

$$\langle n|\mu\rangle = \frac{|\mu|^{(2j+k-1)/2} \mu^n}{\sqrt{\Gamma(n+1)\Gamma(2j+k+n)I_{2j+k-1}(2|\mu|)}} \quad (4.10)$$

as well as

$$|n\rangle = \int d^2\mu f_n(\mu) |\mu\rangle \quad (4.11)$$

where

$$f_n(\mu) = \frac{\mu^{*n} e^{-\frac{1}{2}\mu^*\mu}}{2^{n+1}\pi |\mu|^{(2j+k-1)/2}} \sqrt{\frac{\Gamma(2j+k+n)}{\Gamma(n+1)}} I_{2j+k-1}(2|\mu|) \quad (4.12)$$

Using eq.(4.4), we eventually get the formal expression

$$|\tilde{\mu}_I\rangle = (1 - \tilde{\mu}^* \tilde{\mu})^{j+k/2} \sum_{n=0} \tilde{\mu}^n \sqrt{\frac{\Gamma(2j+k+n)}{\Gamma(n+1)\Gamma(2j+k)}} \int d^2\mu f_n(\mu) |\mu\rangle \quad (4.13)$$

which is the desired result.

5 Conclusion

We studied the properties of exact (all level k) quantum coherent states in the context of Kac -Moody algebras (WZWN models).

Quantum coherent states in the context of string theory on a group manifold (WZWN models) are important since they may help solving the unitarity problem: Having positive norm (no ghost-states appear in the string spectrum), they consistently include the high massive strings (which otherwise are excluded by the spin-level condition).

Coherent states admit (at least) two alternative definitions: (i) as the exponential of the creation operator acting on the ground state, and (ii) as eigenstates of the annihilation operator. In the $k \rightarrow \infty$ limit, all the known properties of usual coherent states are recovered.

In ordinary quantum mechanics and string theory in flat space time (with the usual commutator algebra of harmonic oscillators), the two alternative definitions (i) and (ii) are equivalent. This is not the case in the context of Kac-Moody algebras as in the WZWN models.

In this paper we have constructed coherent states as defined by (ii), compared them to the states (i) we previously constructed, and computed the uncertainty relation in this context. Modification to the Heisenberg relation is given by $2\mathcal{H}/k$ where \mathcal{H} is connected to the string energy. Coherent states (ii) are generally much easier to work with and satisfy automatically and for any k the minimal uncertainty condition. They are thus quassiclassical, in some sense more classical than states (i) which only satisfy it in the $k \rightarrow \infty$ limit.

The coherent states (ii) reduce, for $k = 0$, to the group-related $SU(1,1)$ coherent states constructed in [25], as explained in the introduction. In the opposite limit, for $k \rightarrow \infty$, they reduce to the standard canonical coherent states [1].

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